



Structure-Preserving Finite Element Schemes for Curvature-Driven Flows on Evolving Surfaces: A Review of Methods and Analysis

Diego Gauthier^{1*}, Andres Eriksson², Keisha Souza³

¹Department of Mathematics, Nile Delta Graduate Institute of Mathematics, Sweden

²Department of Mathematics, Douro Institute of Mathematics, Portugal

³Department of Mathematics, Anatolian School of Mathematics, Slovakia

Article Info

Volume 1, Issue 1, 2026

[doi: 10.62715/ijcmg.2026.v1.i1.pp1-6](https://doi.org/10.62715/ijcmg.2026.v1.i1.pp1-6)

Received: 06 November 2025

Accepted: 20 January 2026

Published: 28 March 2026

Abstract

Curvature-driven flows such as mean curvature flow, surface diffusion and Willmore flow move a surface according to its own geometry, and each of them carries a structure that a numerical scheme may or may not inherit: an energy that decays, a volume that is conserved, a parametrization that stays regular. This narrative review surveys finite element schemes for these flows on evolving triangulated surfaces with that question in view. It begins with the parametric method of Dziuk, whose semi-implicit form reproduces the decrease of area, and follows three lines of later work: weak formulations that leave the tangential velocity free so that the mesh distributes itself, fully discrete schemes that conserve enclosed volume and dissipate energy exactly, and the convergence theory that became available once the normal vector and the mean curvature were treated as evolving unknowns. Fourth-order flows, partial differential equations posed on moving surfaces, and the level set alternative are treated more briefly. The review closes with open problems: convergence proofs for the schemes with the best mesh behaviour, structure preservation at higher order in time, and reliable computation through singularities and topological change.

Keywords: Mean curvature flow, Surface diffusion, Willmore flow, Parametric finite elements, Evolving surface finite element method, Energy stability

© 2026 Diego Gauthier et al. This is an open access article under the CC BY license (<https://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

1. Introduction

A geometric evolution equation prescribes the velocity of a surface in terms of the surface itself. In mean curvature flow the normal velocity equals the mean curvature, so that the surface moves in the direction that reduces its area fastest. In surface diffusion the normal velocity is the surface Laplacian of the mean curvature, a law derived by Mullins (1957) for the grooving of heated polycrystals. In Willmore flow the surface descends the integral of the squared mean curvature, the bending energy that Helfrich (1973) placed at the centre of lipid membrane models. These equations appear in grain growth, thin film dewetting, image processing and biomembrane mechanics, and they are nonlinear and degenerate in

* Corresponding author: Diego Gauthier, Department of Mathematics, Nile Delta Graduate Institute of Mathematics, Sweden. E-mail: diego.gauthier@nile.edu

© 2026 Diego Gauthier et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

ways that make their discretization delicate.

The delicacy has a specific source. A parametrized surface can move tangentially without changing shape, so the equations determine only the normal part of the velocity and are not parabolic in the usual sense. A finite element mesh that follows the flow naively may therefore tangle or collapse long before the surface it represents becomes singular. At the same time each flow has a qualitative signature. Area decreases under mean curvature flow, enclosed volume is conserved and area decreases under surface diffusion, and the Willmore energy decays under Willmore flow. A scheme that loses these properties can produce plausible pictures of the wrong evolution.

The phrase structure-preserving is used here for schemes that retain such properties exactly at the discrete level, in the spirit of geometric numerical integration for ordinary differential equations (Hairer et al., 2006). This article reviews how far that programme has been carried for finite element approximations of curvature-driven flows on evolving surfaces, what has been proved about the resulting schemes, and where the theory still falls short of the practice.

2. Scope and Approach

This is a narrative review of the published literature, organized by the structure that each family of schemes is designed to keep. It concentrates on parametric methods, in which the nodes of a triangulation carry the surface, because that is where the question of evolving meshes arises. The two earlier surveys in *Acta Numerica* remain the standard entry points: Deckelnick et al. (2005) for geometric evolution equations in parametric, graph, level set and phase field form, and Dziuk and Elliott (2013) for finite elements for partial differential equations on surfaces. The more recent handbook chapter of Barrett et al. (2020) covers the parametric approach in detail. The present article is shorter and narrower than any of these, and it does not attempt to list every contribution.

3. Flows and the Structures Worth Preserving

All three flows are gradient flows, and the structure of each follows from the energy and the inner product involved. Mean curvature flow is the gradient flow of area with respect to the L^2 inner product of normal velocities. Its smooth theory is well developed: a compact, uniformly convex surface contracts to a point and becomes round as it does so (Huisken, 1984), while non-convex surfaces can pinch off in finite time. Surface diffusion is the gradient flow of area in a weaker inner product of H^{-1} type, which is why it conserves the enclosed volume while still reducing area. Willmore flow is the L^2 gradient flow of the bending energy, and like surface diffusion it is of fourth order in the parametrization.

Table 1 sets the three flows side by side with the properties a discretization would ideally inherit and the sources reviewed below. Figure 1 groups the same properties into four families: energy laws, conserved quantities, mesh structure and the ingredients of the convergence theory. The fourth family is not a property of the continuous flow at all, but experience has shown that the unknowns chosen to make a convergence proof possible also change what the scheme preserves, so it belongs in the same picture.

4. Parametric Finite Elements and the Decrease of Area

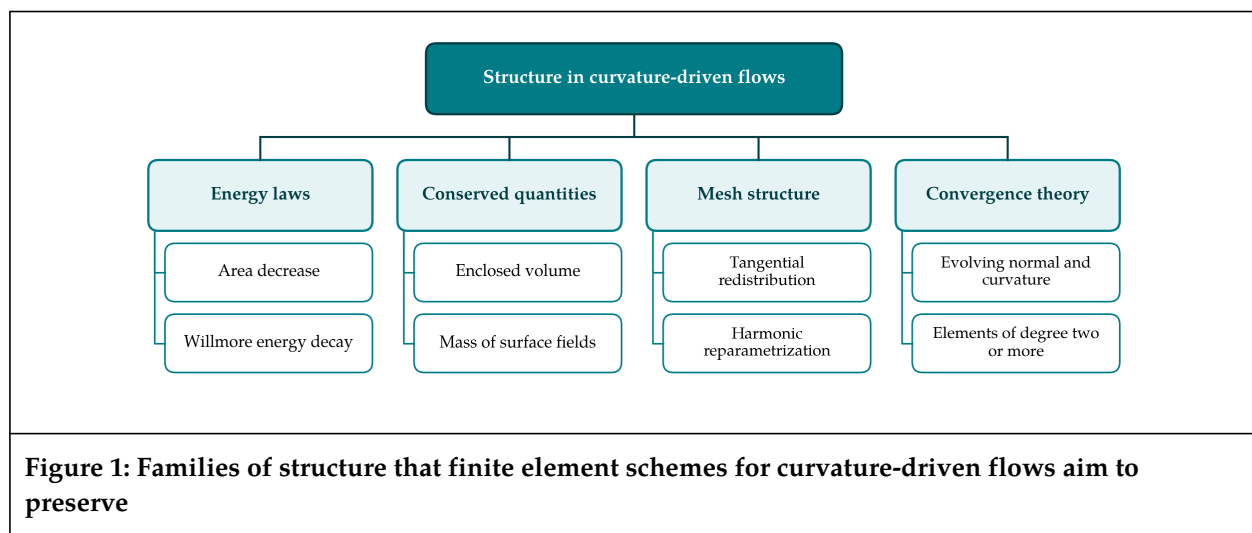
The parametric finite element method for mean curvature flow goes back to Dziuk (1991). It rests on the identity that the Laplace-Beltrami operator applied to the position vector of a surface gives the mean curvature vector, which turns the flow into a heat equation for the position, posed on the surface that the position itself defines. Dziuk had already shown how to discretize the Laplace-Beltrami operator with piecewise linear elements on a triangulated surface (Dziuk, 1988). The evolution scheme assembles the mass and stiffness matrices on the current triangulation and solves one linear system for the new nodal positions, so that the geometry is treated explicitly and the position implicitly.

This semi-implicit splitting has a consequence that was recognized early and has shaped the field. Because the Dirichlet energy of the new position over the old surface bounds the new area from above, the discrete solution satisfies an area inequality at every time step without any restriction on the step size. The scheme is thus unconditionally stable in the energy that defines the flow, and the property is obtained from the form of the discretization and not by adding a stabilizing term. Much of the later literature can

be read as an effort to obtain the same kind of statement for other flows and other quantities.

Table 1: Curvature-driven flows, the structure of each, and representative finite element schemes

Flow	Normal velocity	Structure to preserve	Representative schemes
Mean curvature flow	Mean curvature	Area decreases	Dziuk (1991) ; Kovács et al. (2019)
Surface diffusion	Surface Laplacian of mean curvature	Volume conserved, area decreases	Bänsch et al. (2005) ; Bao and Zhao (2021)
Willmore flow	First variation of bending energy	Bending energy decreases	Rusu (2005) ; Dziuk (2008)
Any of the above	Tangential part not prescribed	Mesh regularity	Barrett et al. (2008) ; Elliott and Fritz (2017)
Flow coupled to a surface field	Curvature plus forcing	Mass of the transported field	Dziuk and Elliott (2007)



What the scheme does not control is the mesh. The nodes move in the normal direction of the discrete surface, and nothing prevents triangles from degenerating as the surface shrinks unevenly. Nor did the energy inequality lead to an error estimate. Convergence was established for curves and for graphs in the 1990s, for instance in the semi-discrete analysis of curve shortening by [Dziuk \(1994\)](#), but for closed surfaces a proof resisted every attempt for close to three decades. A convergence proof for Dziuk's linearly implicit method in the case of curves was eventually given by [Li \(2020\)](#).

5. Mesh Quality as a Structure

A second line of work treats the regularity of the mesh as something to be built into the scheme. The formulation of Barrett, Garcke and Nürnberg tests the velocity only against normal directions, so that the weak form fixes the normal velocity and leaves the tangential velocity to be determined by the discrete curvature equation ([Barrett et al., 2007](#); [Barrett et al., 2008](#)). For curves the resulting semi-discrete solutions are equidistributed: neighbouring segments have equal length wherever they are not parallel. For surfaces no such simple statement holds, but the triangulations remain well shaped in practice over long times, the schemes are unconditionally stable, and the same formulation carries over to surface diffusion and other fourth-order laws with few changes.

A different route modifies the continuous problem. [Elliott and Fritz \(2017\)](#) apply the DeTurck trick, coupling the curvature flow to a harmonic map heat flow that fixes the parametrization. The reparametrized system is strictly parabolic, which removes the degeneracy that obstructs the analysis and at the same time drives the mesh towards a harmonic, hence well-distributed, configuration. Earlier,

Bänsch et al. (2005) had combined a parametric finite element method for surface diffusion with explicit mesh regularization and time step control. The common lesson is that tangential motion is a degree of freedom to be used, and that using it inside the variational formulation is more robust than repairing the mesh afterwards.

6. Exact Conservation and Dissipation After Full Discretization

For surface diffusion the two defining properties pull in different directions once time is discretized. A semi-discrete parametric scheme can conserve the enclosed volume exactly and decrease the area, but a linearly implicit time discretization generally keeps only the area inequality, and the volume drifts by an amount that depends on the time step. For long simulations of dewetting or of sintering this drift matters, since the equilibrium shape is fixed by the volume.

Bao and Zhao (2021) removed the drift for closed curves by evaluating the normal vector not on the old polygon but as a suitable average over the old and the new one. With that choice the change in enclosed area telescopes exactly, and the unconditional energy stability of the parametric formulation is retained. The price is a nonlinear system at each step, solved by a few Newton or fixed-point iterations. Jiang and Li (2021) reached the same pair of properties for curves by a different construction. The device resembles the discrete gradient methods of geometric integration, in which the vector field is evaluated at a carefully chosen intermediate state so that an invariant is kept exactly (Hairer et al., 2006).

7. Convergence Through Evolving Geometric Unknowns

The convergence question for closed surfaces was settled by a change of unknowns. Along mean curvature flow the normal vector and the mean curvature satisfy parabolic evolution equations of their own, derived by Huisken (1984). Kovács et al. (2019) discretize these equations together with the velocity law, using evolving surface finite elements of polynomial degree at least two in space and linearly implicit backward difference formulae in time. The normal and the curvature are then independent finite element functions and are no longer computed from the discrete surface, which gives the stability analysis the control of the geometry that the original scheme lacked. Optimal-order error bounds in the H^1 norm follow for as long as the exact flow remains smooth.

The same strategy has been extended to Willmore flow (Kovács et al., 2021). Its cost is twofold. The system is larger, and the properties that came for free in the position-based schemes are no longer automatic: the area inequality is not built in, and the nodes again move essentially along normals, so that mesh quality has to be restored by an added tangential velocity. The analysis also relies on quadratic or higher elements, for which the geometric error of the interpolated surface is small enough (Demlow, 2009). Proven convergence and built-in structure are therefore available today largely in different schemes.

8. Fourth-Order Flows and Equations on Moving Surfaces

Willmore flow adds the difficulty that the energy involves curvature, so that its first variation contains second derivatives of the curvature and a cubic term. Rusu (2005) introduced a mixed formulation with position and curvature vector as unknowns, and Dziuk (2008) gave a variant, built on a different form of the first variation, that behaves well for parametric surfaces. Reproducing the decay of the bending energy at the fully discrete level is harder than reproducing the decrease of area, and for most schemes in use it is observed in computations and not proved.

In many applications the surface carries a field, such as a surfactant concentration, that is transported with it and feeds back into the velocity. The evolving surface finite element method of Dziuk and Elliott (2007) discretizes the weak form of a conservation law on a moving triangulation. Its nodal basis functions have vanishing material derivative, and as a result the total mass of the transported quantity is conserved by construction. When the surface velocity is itself given by a curvature law, this conservative transport has to be combined with one of the geometric schemes above, and the structure of the coupled energy becomes the object to preserve (Dziuk and Elliott, 2013).

9. The Level Set Alternative

Parametric methods stop when the surface changes topology. The level set method of [Osher and Sethian \(1988\)](#) represents the surface as a level set of a function on a fixed grid and passes through pinch-off and merging without special treatment, and the viscosity solution theory developed for mean curvature flow by [Evans and Spruck \(1991\)](#) gives the evolution a meaning beyond singularities. The trade is an extra space dimension, a degenerate equation and a less direct handle on quantities such as enclosed volume. Structure preservation in the sense of this review has been pursued much further for parametric schemes, and the two approaches are best seen as complementary ([Deckelnick et al., 2005](#)).

10. Gaps and Future Work

The first gap is the one already noted. The schemes with the best mesh behaviour and exact discrete energy laws are, for surfaces, still without convergence proofs, while the schemes with proofs need auxiliary devices to keep their meshes usable. An error analysis of the tangential-velocity-free formulations for surfaces, or a provably convergent scheme that inherits area decrease and good mesh distribution together, would close the most visible divide in the subject.

The second gap concerns time. Exact volume conservation and unconditional energy dissipation are available at first order, and extending both to second or higher order without losing either remains difficult, as it is for gradient flows generally. The third concerns singularities. Every error estimate cited here holds up to the first singular time of the smooth flow, yet computations are routinely carried close to pinch-off, where neither the mesh nor the theory is secure. Finally, anisotropic surface energies, contact lines and triple junctions bring boundary and compatibility conditions whose discrete structure is only partly understood ([Barrett et al., 2020](#)).

11. Conclusion

Three decades after Dziuk's algorithm, finite element schemes for curvature-driven flows can decrease the right energy at every step, conserve enclosed volume to rounding error, keep their meshes regular without remeshing and, in a separate family, converge at provable rates. No single scheme yet does all of this for surfaces in three dimensions. The literature suggests that the choice of unknowns and of the weak formulation decides which structures survive discretization, and that the remaining task is to combine choices that have so far been made separately.

Conflict of interest

To be declared by the author(s) before publication.

Funding

To be declared by the author(s) before publication.

References

- Bao, W. and Zhao, Q. (2021). [A structure-preserving parametric finite element method for surface diffusion](#). *SIAM Journal on Numerical Analysis*, 59(5): 2775-2799.
- Barrett, J.W., Garcke, H. and Nürnberg, R. (2007). [A parametric finite element method for fourth order geometric evolution equations](#). *Journal of Computational Physics*, 222(1): 441-467.
<https://doi.org/10.1016/j.jcp.2006.07.026>
- Barrett, J.W., Garcke, H. and Nürnberg, R. (2008). [On the parametric finite element approximation of evolving hypersurfaces in \$\mathbb{R}^3\$](#) . *Journal of Computational Physics*, 227(9): 4281-4307.
<https://doi.org/10.1016/j.jcp.2007.11.023>
- Barrett, J.W., Garcke, H. and Nürnberg, R. (2020). [Parametric finite element approximations of curvature-driven interface evolutions](#). *Handbook of Numerical Analysis*, 21: 275-423.
- Bänsch, E., Morin, P. and Nochetto, R.H. (2005). [A finite element method for surface diffusion: The](#)

- parametric case. *Journal of Computational Physics*, 203(1): 321-343.
- Deckelnick, K., Dziuk, G. and Elliott, C.M. (2005). *Computation of geometric partial differential equations and mean curvature flow. Acta Numerica*, 14: 139-232. <https://doi.org/10.1017/S0962492904000224>
- Demlow, A. (2009). Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces. *SIAM Journal on Numerical Analysis*, 47(2): 805-827.
- Dziuk, G. (1988). Finite elements for the Beltrami operator on arbitrary surfaces. *Partial Differential Equations and Calculus of Variations*, Lecture Notes in Mathematics 1357, Springer, Berlin, 142-155.
- Dziuk, G. (1991). An algorithm for evolutionary surfaces. *Numerische Mathematik*, 58: 603-611. <https://doi.org/10.1007/BF01385643>
- Dziuk, G. (1994). Convergence of a semi-discrete scheme for the curve shortening flow. *Mathematical Models and Methods in Applied Sciences*, 4(4): 589-606.
- Dziuk, G. (2008). Computational parametric Willmore flow. *Numerische Mathematik*, 111: 55-80.
- Dziuk, G. and Elliott, C.M. (2007). Finite elements on evolving surfaces. *IMA Journal of Numerical Analysis*, 27(2): 262-292. <https://doi.org/10.1093/imanum/drl023>
- Dziuk, G. and Elliott, C.M. (2013). Finite element methods for surface PDEs. *Acta Numerica*, 22: 289-396. <https://doi.org/10.1017/S0962492913000056>
- Elliott, C.M. and Fritz, H. (2017). On approximations of the curve shortening flow and of the mean curvature flow based on the DeTurck trick. *IMA Journal of Numerical Analysis*, 37(2): 543-603.
- Evans, L.C. and Spruck, J. (1991). Motion of level sets by mean curvature. I. *Journal of Differential Geometry*, 33(3): 635-681.
- Hairer, E., Lubich, C. and Wanner, G. (2006). *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*. 2nd edition. Springer, Berlin.
- Helfrich, W. (1973). Elastic properties of lipid bilayers: Theory and possible experiments. *Zeitschrift für Naturforschung C*, 28(11-12): 693-703.
- Huisken, G. (1984). Flow by mean curvature of convex surfaces into spheres. *Journal of Differential Geometry*, 20(1): 237-266.
- Jiang, W. and Li, B. (2021). A perimeter-decreasing and area-conserving algorithm for surface diffusion flow of curves. *Journal of Computational Physics*, 443: 110531.
- Kovács, B., Li, B. and Lubich, C. (2019). A convergent evolving finite element algorithm for mean curvature flow of closed surfaces. *Numerische Mathematik*, 143: 797-853. <https://doi.org/10.1007/s00211-019-01074-2>
- Kovács, B., Li, B. and Lubich, C. (2021). A convergent evolving finite element algorithm for Willmore flow of closed surfaces. *Numerische Mathematik*, 149: 595-643.
- Li, B. (2020). Convergence of Dziuk's linearly implicit parametric finite element method for curve shortening flow. *SIAM Journal on Numerical Analysis*, 58(4).
- Mullins, W.W. (1957). Theory of thermal grooving. *Journal of Applied Physics*, 28(3): 333-339. <https://doi.org/10.1063/1.1722742>
- Osher, S. and Sethian, J.A. (1988). Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations. *Journal of Computational Physics*, 79(1): 12-49. [https://doi.org/10.1016/0021-9991\(88\)90002-2](https://doi.org/10.1016/0021-9991(88)90002-2)
- Rusu, R.E. (2005). An algorithm for the elastic flow of surfaces. *Interfaces and Free Boundaries*, 7(3): 229-239.